

# QUANTUM HALL EFFECT AND NONCOMMUTATIVE GEOMETRY

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**ABSTRACT.** We study magnetic Schrödinger operators with random or almost periodic electric potentials on the hyperbolic plane, motivated by the quantum Hall effect (QHE) in which the hyperbolic geometry provides an effective Hamiltonian. In addition we add some refinements to earlier results. We derive an analogue of the Connes-Kubo formula for the Hall conductance via the quantum adiabatic theorem, identifying it as a geometric invariant associated to an algebra of observables that turns out to be a crossed product algebra. We modify the Fredholm modules defined in [4] in order to prove the integrality of the Hall conductance in this case.

## INTRODUCTION

In [4], continuous and discrete magnetic Hamiltonians containing terms arising from a background hyperbolic geometry were introduced. These may be thought of as effective Hamiltonians for an analogue of the quantum Hall effect studied in a Euclidean model by Bellissard [2] and Xia [20]. We interpret these Hamiltonians, following a suggestion of Bellissard, as modelling spinless electrons in a conducting material with a perturbation term arising from a background hyperbolic geometry. (In [4] we took the somewhat different view that the conducting material exhibited hyperbolic geometry.) They motivate constructing Fredholm modules associated in a natural way with Riemann surfaces and two dimensional orbifolds which give a higher genus analogue of the work of Bellissard (which is the genus one case) on the quantum Hall effect. In [4] we considered Hamiltonians invariant under a projective action of a Fuchsian group  $\Gamma$ . We will only discuss groups whose actions on hyperbolic space are free here and refer the reader to [12] for the more general case. In this paper we allow in addition a random potential (which may be thought of as modelling impurities) so that the invariance of the Hamiltonian under the Fuchsian group is replaced by a type of ergodicity assumption. There is an analogue of the Connes-Kubo cocycle of the Euclidean case for the Hall conductance. This cocycle takes values which are integer multiples of a fundamental unit in the case of free actions and rational multiples for non-free actions. Integrality follows by showing

that the cocycle gives the index of a certain Fredholm operator (the conductance may also be thought of in terms of a topological index). Thus the models in [4, 12] fit the noncommutative geometry framework for magnetic Hamiltonians (see [8]).

We begin by reviewing the construction of magnetic Hamiltonians in a continuous model with a background hyperbolic geometry term. There are also discrete versions which are generalised Harper operators [19, 4, 5, 12]. Our model of hyperbolic space is the upper half-plane  $\mathbb{H}$  in  $\mathbb{C}$  equipped with its usual Poincaré metric  $(dx^2 + dy^2)/y^2$ , and symplectic area form  $\omega_{\mathbb{H}} = dx \wedge dy/y^2$ . The group  $\mathbf{PSL}(2, \mathbb{R})$  acts transitively on  $\mathbb{H}$  by Möbius transformations

$$x + iy = \zeta \mapsto g\zeta = \frac{a\zeta + b}{c\zeta + d}, \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Any Riemann surface of genus  $g$  greater than 1 can be realised as the quotient of  $\mathbb{H}$  by the action of its fundamental group realised as a cocompact torsion-free discrete subgroup  $\Gamma$  of  $\mathbf{PSL}(2, \mathbb{R})$ .

Pick a 1-form  $\eta$  such that  $d\eta = \theta\omega_{\mathbb{H}}$ , for some fixed  $\theta \in \mathbb{R}$ . As in geometric quantisation we may regard  $\eta$  as defining a connection  $\nabla = d - i\eta$  on a line bundle  $\mathcal{L}$  over  $\mathbb{H}$ , whose curvature is  $\theta\omega_{\mathbb{H}}$ . Physically we can think of  $\eta$  as the electromagnetic vector potential for a uniform magnetic field of strength  $\theta$  normal to  $\mathbb{H}$ . Using the Riemannian metric the Hamiltonian of an electron in this field is given in suitable units by

$$H = H_{\eta} = \frac{1}{2}\nabla^*\nabla = \frac{1}{2}(d - i\eta)^*(d - i\eta).$$

Comtet [6] has shown that  $H$  differs from a multiple of the Casimir element for  $\mathbf{PSL}(2, \mathbb{R})$ ,  $\frac{1}{8}\mathbf{J}.\mathbf{J}$ ,  $J_1$ ,  $J_2$  and  $J_3$  denote a certain representation of generators of the Lie algebra  $sl(2, \mathbb{R})$ , satisfying  $[J_1, J_2] = -iJ_3$ ,  $[J_2, J_3] = iJ_1$ ,  $[J_3, J_1] = iJ_2$ , so that  $\mathbf{J}.\mathbf{J} = J_1^2 + J_2^2 + J_3^2$  is the quadratic Casimir element showing the underlying  $\mathbf{PSL}(2, \mathbb{R})$ -invariance of the theory. Comtet has computed the spectrum of the unperturbed Hamiltonian  $H_{\eta}$ , for  $\eta = -\theta dx/y$ , to be the union of finitely many eigenvalues  $\{(2k+1)\theta - k(k+1) : k = 0, 1, 2, \dots < \theta - \frac{1}{2}\}$ , and the continuous spectrum  $[\frac{1}{4} + \theta^2, \infty)$ . Any  $\eta$  is cohomologous to  $-\theta dx/y$  (since they both have  $\omega_{\mathbb{H}}$  as differential) and forms differing by an exact form  $d\phi$  give equivalent models: in fact, multiplying the wave functions by  $\exp(i\phi)$  shows that the models for  $\eta$  and  $-\theta dx/y$  are unitarily equivalent. This equivalence also intertwines the  $\Gamma$ -actions so that the spectral densities for the two models also coincide.

This Hamiltonian can be perturbed by adding a potential term  $V$ . In [4], we took  $V$  to be invariant under  $\Gamma$ . In [5] we allowed any smooth random potential function  $V$  on  $\mathbb{H}$  using two general notions of random potential (in the literature random usually refers to the  $\Gamma$ -action on the disorder space being required to admit an ergodic invariant measure). The class of random

potentials we consider here contains any smooth bounded potential  $V$ . The perturbed Hamiltonian  $H_{\eta,V} = H_{\eta} + V$  has unknown spectrum for such general  $V$ . However we are able to deduce some qualitative aspects of the spectrum of these Hamiltonians by using a reduction (via Morita equivalence) to a simpler case: that of a discrete model.

In Section 2, we extend the hyperbolic Connes-Kubo formula for the Hall conductance for the continuous model in [4], to the non-periodic case. We show that this hyperbolic Connes-Kubo cocycle is cohomologous to another cyclic 2-cocycle which is the Chern character of a Fredholm module, from which we can deduce that the Hall conductance takes on integral values in  $2(g-1)\mathbb{Z}$  ( $g > 1$  being the genus). This result has been generalized in [12] where for general cocompact Fuchsian groups  $\Gamma$ , it is shown that the conductance takes on values in  $\phi\mathbb{Z}$ , where  $\phi$  denotes the orbifold Euler characteristic of the orbifold  $\mathbb{H}/\Gamma$ , i.e. the conductance can take on certain fractional values. In the Appendix we give a derivation of the hyperbolic Connes-Kubo formula for the Hall conductance, using the quantum adiabatic theorem and standard physical reasoning.

## 1. CONTINUOUS MODEL

**1.1. The geometry of the hyperbolic plane.** The upper half-plane can be mapped by the Cayley transform  $z = (\zeta - i)/(\zeta + i)$  to the unit disc  $\mathbb{D}$  equipped with the metric  $|dz|^2/(1-|z|^2)^2$  and symplectic form  $dz d\bar{z}/2i(1-|z|^2)^2$ , on which  $\mathbf{PSU}(1,1)$  acts, and some calculations are more easily done in that setting. In order to preserve flexibility we shall work more abstractly with a Lie group  $G$  acting transitively on a space  $X \sim G/K$ . Although we shall ultimately be interested in the case of  $G = \mathbf{PSL}(2, \mathbb{R})$  or  $\mathbf{PSU}(1,1)$ , and  $K$  the maximal compact subgroup which stabilises  $\zeta = i$  or  $z = 0$  so that  $X = \mathbb{H}$  or  $X = \mathbb{D}$ , those details will play little role in many of our calculations, though we shall need to assume that  $X$  has a  $G$ -invariant Riemannian metric and symplectic form  $\omega_X$ . We shall denote by  $\Gamma$  a discrete subgroup of  $G$  which acts freely on  $X$  and hence intersects  $K$  trivially.

We shall assume that  $\mathcal{L}$  is a hermitian line bundle over  $X$ , with a connection,  $\nabla$ , or equivalently, for each pair of points  $w$  and  $z$  in  $X$ , we denote by  $\tau(z, w)$  the parallel transport operator along the geodesic from  $\mathcal{L}_w$  to  $\mathcal{L}_z$ . In  $\mathbb{H}$  with the line bundle trivialised and  $\eta = \theta dx/y$  one can calculate explicitly that

$$\tau(z, w) = \exp\left(i \int_w^z \eta\right) = [(z - \bar{w})/(w - \bar{z})]^\theta.$$

For general  $\eta$  we have  $\eta - \theta dx/y = d\phi$  and

$$\tau(z, w) = \exp\left(i \int_w^z \eta\right) = [(z - \bar{w})/(w - \bar{z})]^\theta \exp(i(\phi(z) - \phi(w))).$$

Parallel transport round a geodesic triangle with vertices  $z, w, v$ , gives rise to a holonomy factor:

$$\varpi(v, w, z) = \tau(v, z)^{-1} \tau(v, w) \tau(w, z),$$

and this is clearly the same for any other choice of  $\eta$ , so we may as well work in the general case.

**Lemma 1.1.** *The holonomy can be written as  $\varpi(v, w, z) = \exp(i\theta \int_{\Delta} \omega_{\mathbb{H}})$ , where  $\Delta$  denotes the geodesic triangle with vertices  $z, w$  and  $v$ . The holonomy is invariant under the action of  $G$ , that is  $\varpi(v, w, z) = \varpi(gv, gw, gz)$ , and under cyclic permutations of its arguments. Transposition of any two vertices inverts  $\varpi$ . For any four points  $u, v, w, z$  in  $X$  one has*

$$\varpi(u, v, w) \varpi(u, w, z) = \varpi(u, v, z) \varpi(v, w, z).$$

**1.2. Algebra of observables and random or almost periodic potentials.** The algebra of physical observables that we consider in the continuous model should include the operators  $f(H_{\eta, V})$  for any bounded continuous function  $f$  on  $\mathbb{R}$  and for any smooth random potential function  $V$  on  $\mathbb{H}$  with disorder space  $\Omega$ . We will see that the twisted  $C^*$ -algebra of the groupoid  $\mathcal{G} = \Gamma \backslash (X \times X \times \Omega)$ , twisted by  $\varpi$ , is large enough to contain all such operators. This algebra also turns out to be the twisted  $C^*$ -algebra of the foliation  $\Omega_{\Gamma}$ . This  $C^*$ -algebra is strongly Morita equivalent to the cross product  $C^*$ -algebra  $C(\Omega) \rtimes_{\sigma} \Gamma$ , where  $\sigma$  is a multiplier on  $\Gamma$  which is determined by  $\varpi$ .

**Assumptions** The *disorder space*  $\Omega$  we assume to be compact, to admit a Borel probability measure  $\Lambda$ ; and that there is a continuous action of  $\Gamma$  on  $\Omega$  with a dense orbit.

The geometrical data described in the last subsection enables us to easily describe the first of the two  $C^*$ -algebras which appear in the theory. This twisted algebra of kernels, which was introduced by Connes [8] is the  $C^*$ -algebra  $\mathcal{B}$  generated by compactly supported smooth functions on  $X \times X \times \Omega$  with the multiplication

$$k_1 * k_2(z, w, r) = \int_X k_1(z, v, r) k_2(v, w, r) \varpi(z, w, v) dv,$$

(where  $dv$  is the  $G$ -invariant measure defined by the metric) and  $k^*(z, w, r) = \overline{k(w, z, r)}$ . The trace on  $\mathcal{B}$  is given by,  $\tau_{\mathcal{B}}(k) = \int_{X \times \Omega} k(z, z, r) dz d\Lambda(r)$ . Observe that  $X \times X \times \Omega$  is a groupoid with space of units  $X \times \Omega$  and with source and range maps  $s((z, w, r)) = (w, r)$  and  $r((z, w, r')) = (z, r')$ . Then the algebra of twisted kernels is the extension of the  $C^*$ -algebra of the groupoid  $X \times X \times \Omega$  defined by the cocycle  $((v, w, r), (w, z, r)) \mapsto \varpi(v, w, z)$ , [16].

**Lemma 1.2.** *The algebra  $\mathcal{B}$  has a representation  $\pi$  on the space  $\mathcal{H}$  of  $L^2$  sections of  $\mathcal{L} \rightarrow X \times \Omega$  defined by*

$$(\pi(k)\psi)(z, r) = \int_X k(z, w, r) \tau(z, w) \psi(w, r) dw.$$

We now pick out a  $\Gamma$ -invariant subalgebra  $\mathcal{B}^\Gamma$  of  $\mathcal{B}$ . This condition reduces simply to the requirement that the kernel satisfies  $k(\gamma^{-1}z, \gamma^{-1}w, \gamma^{-1}r) = k(z, w, r)$  for all  $\gamma \in \Gamma$ . As before, observe that  $\Gamma \backslash (X \times X \times \Omega)$  is a groupoid whose elements are  $\Gamma$  orbits  $(x, y, v)_\Gamma = \{(\gamma x, \gamma y, \gamma v) : \gamma \in \Gamma\}$ , with source and range maps  $s((x, y, v)_\Gamma) = (y, v)$  and  $r((x, y, v)_\Gamma) = (x, v)$ . The space of units is  $\Omega_\Gamma = \Gamma \backslash (X \times \Omega)$ . Then the algebra of invariant twisted kernels  $\mathcal{B}^\Gamma$  is the extension of the  $C^*$ -algebra of the groupoid  $\Gamma \backslash (X \times X \times \Omega)$  defined by the cocycle  $((v, w, r), (w, z, r)) \mapsto \varpi(v, w, z)$ , [16]. With our assumptions on the disorder space  $\Omega$ , there is in general *no* trace on the algebra  $\mathcal{B}^\Gamma$ , and there may not even be a weight on this algebra in general. However, we mention that under the additional assumption that the measure  $\Lambda$  on  $\Omega$  is  $\Gamma$ -invariant, the natural trace  $\tau_{\mathcal{B}^\Gamma}$  for this algebra is given by the same formula as before except that the integration is now over  $\Omega_\Gamma = \Gamma \backslash (X \times \Omega)$  rather than  $X \times \Omega$ , where we have identified  $\Omega_\Gamma$  with a fundamental domain:  $\tau_{\mathcal{B}^\Gamma}(T) = \int_{\Omega_\Gamma} T(z, z, r) dz d\Lambda(r)$ . We also mention that under the additional assumption that the measure  $\Lambda$  on  $\Omega$  is quasi- $\Gamma$ -invariant, the natural tracial weight  $\tau_{\mathcal{B}^\Gamma}$  for this algebra is given by  $\tau_{\mathcal{B}^\Gamma}(T) = \int_{X \times \Omega} f(z, r)^2 T(z, z, r) dz d\Lambda(r)$ , where  $f \in C_c(X \times \Omega)$  is such that  $\sum_{\gamma \in \Gamma} (\gamma^* f)^2 = 1$ .

We now recall a notion due to Connes [8].

**Definition 1.3.** *A random or almost periodic potential on  $X$  is a continuous family of smooth functions on the disorder space,  $\Omega \ni r \mapsto V_r \in C^\infty(X)$  where the following equivariance is imposed:*

$$V_{\gamma r} = \gamma^* V_r \quad \forall \gamma \in \Gamma, \forall r \in \Omega.$$

**Remarks 1.4.** *If  $V$  is a  $\Gamma$ -invariant potential on  $X$ , then it is clearly random for any disorder space. More generally, if  $V$  is a arbitrary smooth function on  $X$  such that the set  $\{\gamma^* V : \gamma \in \Gamma\}$  has compact closure in the strong operator topology in  $B(L^2(X))$ , then  $V$  is a random potential.*

The reason the Hamiltonian can be accommodated within the algebra  $\mathcal{B}^\Gamma$  is not hard to explain. Fix a base point  $u \in \mathbb{D}$  and introduce:

$$\sigma(x, y) = \varpi(u, xu, xyu)$$

$$\phi(z, \gamma) = \varpi(u, \gamma^{-1}u, \gamma^{-1}z) \tau(u, z)^{-1} \tau(u, \gamma^{-1}z).$$

Then  $\sigma$  is the group 2-cocycle in the projective action of  $\mathbf{PSU}(1, 1)$  on  $L^2(\mathbb{D})$  defined by:

$$U(\gamma)\psi(z) = \phi(z, \gamma)\psi(\gamma^{-1}z)$$

where  $\psi \in L^2(\mathbb{D})$ ,  $\gamma \in \mathbf{PSU}(1,1)$ . Note that  $U$  is constructed so that the  $\Gamma$ -invariant algebra  $\pi(\mathcal{B}^\Gamma)$  is the intersection of  $\pi(\mathcal{B})$  with the commutant of  $U$ . Recall that the unperturbed Hamiltonian  $H = H_\eta$  commutes with the projective representation  $U$  (cf. Lemma 4.9, [4]). So we see that  $H$  is affiliated to the von Neumann algebra generated by the representation  $\pi$  of  $\mathcal{B}^\Gamma$  (cf. Corollary 4.2 [4]).

A random potential  $V$  can be viewed as defining an equivariant family of Hamiltonians  $\Omega \ni r \mapsto H_{\eta,V_r} = H + V_r \in \text{Oper}(L^2(X))$  where  $\text{Oper}(L^2(X))$  denotes closed operators on  $L^2(X)$ . Brüning and Sunada have proved an estimate on the Schwartz kernel of the heat operator for any elliptic operator, and in particular for  $\exp(-tH_{\eta,V_r})$  for  $t > 0$ , which implies that it is  $L^1$  in each variable separately. Since this kernel is  $\Gamma$ -equivariant it follows (in exactly the same fashion as Lemma 4 of [3]) that this estimate implies that  $\exp(-tH_{\eta,V_r})$  is actually in the algebra  $\mathcal{B}^\Gamma$ .

**Lemma 1.5.** *One has  $f(H_{\eta,V}) \in \mathcal{B}^\Gamma$  for any bounded continuous function  $f$  on  $\mathbb{R}$  and for any random potential  $V$  on  $X$ . In particular, the spectral projections of  $H_{\eta,V}$  corresponding to gaps in the spectrum lie in  $\mathcal{B}^\Gamma$ .*

Following [2],[15] but using our weaker assumptions we now have the

**Theorem 1.6.** *Let  $V$  be a smooth bounded function on  $X$ . Then  $V$  is a random potential for some disorder space  $\Omega$  and therefore  $f(H_{\eta,V}) \in \mathcal{B}^\Gamma$  for any bounded continuous function  $f$  on  $\mathbb{R}$ .*

*Example.* Let the Iwasawa decomposition of  $\mathbf{PSU}(1,1)$  be written  $KAN$  then  $PSL(2, \mathbb{Z})$  acts on  $\mathbb{D} = \mathbf{PSU}(1,1)/K$  by Möbius transformations so that  $\Gamma \subset PSL(2, \mathbb{Z})$  also acts. Let  $g_{\lambda,w}(z) = \lambda \frac{1-|z|^2}{|w-z|^2}$  where  $\lambda \in \mathbb{R}^+ \cong A$ ,  $w \in U(1) \cong K$  and  $z \in \mathbb{D}$ . Now let  $\gamma = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$  and we calculate

$$U(\gamma)g_{\lambda,w}U(\gamma^{-1}) = g_{\lambda_{\gamma,w}\lambda,\gamma w}$$

where  $\lambda_{\gamma,w} = |\bar{\beta}w + \bar{\alpha}|^{-2}$ . The stabiliser of  $g_{1,1}$  is  $\{\pm \begin{pmatrix} 1-in & in \\ -in & 1+in \end{pmatrix} : n \in \mathbb{R}\}$ . This group is  $MN$  where  $MAN$  is the maximal parabolic subgroup. Thus we have the usual action of  $\mathbf{PSU}(1,1)$  on  $\mathbf{PSU}(1,1)/MN$  and hence *a fortiori* a  $\Gamma$ -action which is known to be ergodic, cf. [21]. Note that, regarding  $\{e^{-g_{\lambda,w}}\}$  as a set of bounded multiplication operators on  $L^2(\mathbb{D})$ , the strong closure of  $\{U(\gamma)e^{-g_{\lambda,w}}U(\gamma^{-1}) \mid \lambda \in \mathbb{R}, w \in U(1)\}$  is homeomorphic to  $S^2$ . (This is because taking the strong closure adds the zero and identity operator to the set.) Thus in this example the disorder space is  $S^2$  which admits a dense orbit and a quasi-invariant ergodic probability measure.

**1.3. Morita equivalence.** Our ability to calculate the possible values of our generalised Connes-Kubo cocycle rests on a Morita equivalence argument due initially to [14]. We use the twisted version, [17], [18]. We have already noted that  $\mathcal{B}$  is the  $C^*$ -algebra of an extension of the groupoid  $X \times X \times \Omega$  by a cocycle defined by  $\varpi$ , and  $\Gamma$  invariance of  $\varpi$  means that  $\mathcal{B}^\Gamma$  is likewise the  $C^*$ -algebra of an extension of  $\Gamma \backslash (X \times X \times \Omega)$  by  $\varpi$ , where  $\Gamma \backslash (X \times X \times \Omega)$  denotes the groupoid obtained by factoring out the diagonal action of  $\Gamma$ . More precisely, the groupoid elements are  $\Gamma$  orbits  $(x, y, v)_\Gamma = \{(\gamma x, \gamma y, \gamma v) : \gamma \in \Gamma\}$ , with source and range maps  $s((x, y, v)_\Gamma) = (y, v)$  and  $r((x, y, v)_\Gamma) = (x, v)$ . Therefore  $(x_1, y_1, v_1)_\Gamma$  and  $(x_2, y_2, v_2)_\Gamma$  are composable if and only if  $y_1 = \gamma x_2$  and  $v_1 = \gamma v_2$  for some  $\gamma \in \Gamma$ , and then the composition is  $(x_1, \gamma y_2, \gamma v_2)_\Gamma$ . We also note that  $\Omega \times \Gamma$  is a groupoid. The source and range maps are  $s((v, \gamma)) = \gamma v$  and  $r((v, \gamma)) = v$ . Therefore the elements  $(v_1, \gamma_1)$  and  $(v_2, \gamma_2)$  are composable if and only if  $v_1 = \gamma_2 v_2$ , and the composition is  $(\gamma_2^{-1} v_1, \gamma_1 \gamma_2)$ .

**Theorem 1.7.** *The algebra  $\mathcal{B}^\Gamma$  is Morita equivalent to the twisted cross product algebra  $C(\Omega) \rtimes_{\bar{\sigma}} \Gamma$ .*

The proof is a consequence of:

**Lemma 1.8.** *The line bundle  $\mathcal{L}$  over  $X \times \Omega$  provides an equivalence (in the sense of [17] Definition 5.3) between the groupoid extensions  $(\Gamma \backslash (X \times X \times \Omega))^\varpi$  of  $\Gamma \backslash (X \times X \times \Omega)$  defined by  $\varpi$  and  $(\Omega \times \Gamma)^\sigma$  of  $\Omega \times \Gamma$  defined by  $\bar{\sigma}$ .*

Using the orientation reversing diffeomorphism of the Riemann surface  $\Sigma = \Gamma \backslash X$ , one can show as in Proposition 7 [4] that the algebra  $C(\Omega) \rtimes_{\bar{\sigma}} \Gamma$  is isomorphic to  $C(\Omega) \rtimes_\sigma \Gamma$ , where  $\bar{\sigma}$  denotes the complex conjugate of  $\sigma$ . Morita equivalence of algebras implies their  $K$ -groups are the same. It is possible to calculate the values taken by our cyclic cocycles for the continuous model in terms those taken by explicit cocycles on  $C(\Omega) \rtimes_{\bar{\sigma}} \Gamma$ . The method uses generalisations of arguments first developed for the study of the Baum-Connes conjecture. Full details are in [13] and [4].

**1.4. A hyperbolic Connes-Kubo formula, part I.** The quotient  $\Sigma = \mathbb{H}/\Gamma$  is a Riemann surface when  $\Gamma$  is a cocompact torsion free subgroup of  $\mathbf{PSL}(2, \mathbb{R})$ . On a Riemann surface it is natural to investigate changes in the potential corresponding to adding multiples of the real and imaginary parts of holomorphic 1-forms. (For the genus one case with an imaginary period this amounts to choosing forms whose integral round one sort of cycle vanishes but the integral around the other cycle is non-trivial. Physically this would correspond to putting a non-trivial voltage across one cycle and measuring a current round the other.)

We let  $a_j, j = 1, 2, \dots, 2g$  be a normalized symplectic basis of harmonic 1-forms on  $\Sigma = \mathbb{H}/\Gamma$  where  $a_{j+g} = *a_j, j = 1, 2, \dots, g$ , and  $\int_\Sigma a_j \wedge a_{j+g} = 1$

for all  $j = 1, \dots, g$ . We introduce the map from  $\mathbb{H}$  to  $\mathbb{R}^{2g}$  given by  $\Xi : z \mapsto (\int_u^z a_1, \dots, \int_u^z a_{2g})$ . It is the lift to  $\mathbb{H}$  of the Abel-Jacobi map, [9] (this map is usually regarded as mapping from  $\Sigma_g$  to the Jacobi variety however we are thinking of it as a map between the universal covers of these spaces). Notice that  $\Xi$  gives the period lattice in  $\mathbb{R}^{2g}$  (that is the lattice determined by the periods of the harmonic forms  $a_j$ ) to be the standard integer lattice  $\mathbb{Z}^{2g}$  so that  $J(\Sigma_g) = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ . We give  $\mathbb{R}^{2g}$  the distinguished basis consisting of the vertices in this integer period lattice. We write for the corresponding coordinates  $u_1, u_2, \dots, u_{2g}$ . Let  $\omega_J = \sum_{j=1}^g du_j \wedge du_{j+g}$  denote the symplectic form on  $\mathbb{R}^{2g}$ . The closed 1-forms  $c_j = \Xi^*(du_j)$  are cohomologous to  $a_j$  for all  $j = 1, \dots, 2g$ , and therefore we have

**Lemma 1.9.** *In the notation above,  $\Xi^*(\omega_J)$  is cohomologous to  $\sum_{j=1}^g a_j \wedge a_{j+g}$ .*

Suppose that  $\alpha \in \mathcal{B}$  is a kernel *decaying rapidly*. By this we mean that it satisfies an estimate

$$|\alpha(x, y, r)| \leq \phi(d(x, y)), \quad r \in \Omega,$$

where  $\phi$  is a positive and rapidly decreasing function on  $\mathbb{R}$ . Now define

$$\delta_j \alpha = [\Omega_j, \alpha], \quad \text{i.e.} \quad \delta_j \alpha(x, y, r) = (\Omega_j(x) - \Omega_j(y))\alpha(x, y, r),$$

where  $\Omega_j(z) = i \int_u^z a_j$ . Since  $\Omega_j(\gamma.z) - \Omega_j(z)$  is a constant depending only on  $\gamma$  but independent of  $z$ , and  $|\Omega_j(\gamma.z) - \Omega_j(z)| \leq C \|a_j\|_\infty d(z, \gamma.z) \leq C_j \ell(\gamma)$ , where  $\|a_j\|_\infty$  is the supremum norm of  $a_j$ ,  $\gamma \in \Gamma$ ,  $d(z, \gamma.z)$  is the Riemannian distance between  $z$  and  $\gamma z$ , and  $\ell(\gamma)$  is the word length of  $\gamma$ . It follows that  $\delta_j \alpha$  lies in  $\mathcal{B}$  and therefore  $\delta_j$  is a densely defined derivation on the algebra  $\mathcal{B}$ , and hence also on  $\mathcal{B}^\Gamma$  since clearly if  $\alpha$  is  $\Gamma$ -invariant, then so is  $\delta_j \alpha$ .

We may summarise the previous discussion as

**Lemma 1.10.** *For operators  $A_0, A_1, A_2$  in  $\mathcal{B}^\Gamma$  whose integral kernels are rapidly decaying we have cyclic cocycles defined by*

$$c_{j,k}(A_0, A_1, A_2) = \text{tr}_{\mathcal{B}^\Gamma}(A_0[\delta_j A_1, \delta_k A_2]) = \text{tr}_{\mathcal{B}^\Gamma}(A_0[\Omega_j, A_1][\Omega_k, A_2])$$

for  $j, k = 1, \dots, 2g$ .

The cyclic cocycle  $c_{jk}$  can be interpreted as the Kubo formula for the conductance due to currents in the  $k$  direction induced by electric fields in the  $j$  direction, as explained in the Appendix.

## 2. A FREDHOLM MODULE

We shall now assume that  $X$  has a spin structure, and we write  $\mathcal{S}$  for the spin bundle. The representation of  $\mathcal{B}^\Gamma$  on  $\mathcal{H}$  can then be extended to an



action on  $\mathcal{H} \otimes \mathcal{S}$ . This module can be equipped with a Fredholm structure by taking  $F$  to be Clifford multiplication by a suitable unit vector (to be explained below), and using the product of the trace on  $\mathcal{H}$  and the graded trace on the Clifford algebra. (If  $\varepsilon$  denotes the grading operator on the spinors then the graded trace is just  $\text{tr} \circ \varepsilon$ .)

The same module can also be described more explicitly: it splits into  $\mathcal{H} \otimes \mathcal{S}^+ \oplus \mathcal{H} \otimes \mathcal{S}^-$  (with the superscripted sign indicating the eigenvalue of  $\varepsilon$ ). Suppose that  $\varphi$  is a  $U(1)$  valued function on the group, which satisfies  $\varphi(kgh) = \chi_1(k)\varphi(g)\chi_2(h)$  for  $k$  and  $h$  in  $K$  and some  $\sigma$ -characters  $\chi_1$  and  $\chi_2$  of  $K$ . The involution  $F$  can be taken to be the matrix multiplication operator:  $F = \begin{pmatrix} 0 & \varphi^* \\ \varphi & 0 \end{pmatrix}$ . We may take for  $\varphi$  the function used by Connes [8] which is essentially the Mishchenko element. In the next subsection we will see that the module is 2-summable for suitably decaying kernels. Since  $\varphi$  is invariant under simultaneous conjugation of both variables by elements of  $\Gamma$ ,  $F$  preserves the  $\Gamma$ -invariant subspace.

**Theorem 2.1.** *There is a dense subalgebra  $\mathcal{B}_0^\Gamma$  of  $\mathcal{B}^\Gamma$  stable under the holomorphic functional calculus and a 2-summable Fredholm module  $(F, \mathcal{H} \otimes \mathcal{S})$  for  $\mathcal{B}_0^\Gamma$  with Chern character given by the cyclic 2-cocycle  $\tau_{c,\Gamma}(A_0, A_1, A_2)$  which is equal to*

$$\int_{X_\Gamma \times X \times X} \Phi(z, x, y) \varpi(z, x, y) k_0(z, x, r) k_1(x, y, r) k_2(y, z, r) dz dx dy,$$

$r \in \Omega$ , where the operators  $A_0, A_1, A_2$  are in  $\mathcal{B}_0^\Gamma$ , and whose Schwartz kernels are  $k_0, k_1, k_2$  respectively. Here  $\Phi(z, x, y) = \int_\Delta \omega_\mathbb{H}$  is the oriented hyperbolic area of a geodesic triangle  $\Delta$  with vertices at  $x, y, z$ . Furthermore if  $P(r)$  is a projection into a gap in the spectrum of the Hamiltonian  $H_{\eta,V}$ . Then  $P(r)$  lies in a 2-summable dense subalgebra  $\mathcal{B}_0^\Gamma$  of  $\mathcal{B}^\Gamma$  and for almost any  $r \in \Omega$  one has

$$\text{index}(P(r)FP(r)) = \langle \tau_{c,\Gamma}, [P(r)] \rangle \in 2(g-1)\mathbb{Z}.$$

**2.1. Summability of the Fredholm module.** The technical parts of the proof of the previous theorem rest on a lengthy calculation together with a key estimate on kernels  $k(z, w, r)$  on  $\mathbb{H} \times \mathbb{H} \times \Omega$  which represent smooth functions of the resolvent of  $H + V$ . This estimate has the form

$$|k(z, w, r)|^2 \leq C_2 \exp(-C_3 d(z, w)^2), \quad (**)$$

where  $C_2, C_3$  are constants (note that the RHS is independent of  $r$ ). This estimate is a result of [3]. Since operators with kernels which have support in a band around the diagonal are dense in the algebra  $\mathcal{B}^\Gamma$  so too is the set of operators with kernels satisfying (\*\*). We denote by  $\mathcal{B}_0^\Gamma$  the subalgebra consisting of operators  $A \in \mathcal{B}^\Gamma$ , with  $[F, A]$  a Hilbert-Schmidt operator. Now  $\mathcal{B}_0^\Gamma$  is dense and by [7]  $\mathcal{B}_0^\Gamma$  is stable under the holomorphic functional calculus. The last claim of the corollary on the range of values taken by the

cyclic cocycle follows using Morita equivalence with  $C(\Omega) \rtimes_{\bar{\sigma}} \Gamma$ . The details are in [13][4].

**2.2. The hyperbolic Connes-Kubo formula, part II.** We now have many cyclic 2-cocycles associated to our model. We combine the cyclic 2-cocycles of subsection 1.4 to produce a Connes-Kubo cocycle for the hyperbolic Hall conductance in Proposition 2.2, and our goal is to show that it is cohomologous to the Chern character of the Fredholm module  $\tau_{c,\Gamma}$  as given in Theorem 2.1.

For  $j = 1, \dots, g$ , consider  $\Psi_j(z, x, y)$  which is given by,

$$(\Omega_j(x) - \Omega_j(y))(\Omega_{j+g}(y) - \Omega_{j+g}(z)) - (\Omega_{j+g}(x) - \Omega_{j+g}(y))(\Omega_j(y) - \Omega_j(z)).$$

We claim first that  $\sum_{j=1}^g \Psi_j(z, x, y)$  is proportional to the ‘symplectic area’ of a triangle in  $\mathbb{R}^{2g}$  with vertices  $\Xi(x), \Xi(y), \Xi(z)$ . To prove this it suffices to assume that the base point in  $\mathbb{H}$  is one of the vertices of the triangle, say  $z$ . Consider the expression

$$\sum_{j=1}^g \Psi_j(z, x, y) = \sum_{j=1}^g (\Omega_j(x)\Omega_{j+g}(y) - \Omega_{j+g}(x)\Omega_j(y)).$$

Let  $s$  denote the symplectic form on  $\mathbb{R}^{2g}$  given by:  $s(u, v) = \sum_{j=1}^g (u_j v_{j+g} - u_{j+g} v_j)$ . The ‘symplectic area’ of a triangle  $\Delta_E$  with vertices  $0, \Xi(x), \Xi(y)$  is given by  $s(\Xi(x), \Xi(y))/2$ . To appreciate why this is so we need an argument from [9] (pp 333-336). The form  $s$  is the two form on  $\mathbb{R}^{2g}$  given by

$$\omega_J = \sum_{j=1}^g du_j \wedge du_{j+g}.$$

Now the symplectic area of a triangle  $\Delta_E$  in  $\mathbb{R}^{2g}$  with vertices  $0, \Xi(x), \Xi(y)$  is by definition the integral of  $\omega_J$  over the triangle. A brief calculation reveals that this yields  $s(\Xi(x), \Xi(y))/2$ , proving our claim. We have now established the following result.

**Proposition 2.2.** *The higher genus analogue of the Connes-Kubo formula is given by the cyclic 2-cocycle  $\tau_K$  on  $\mathcal{B}^\Gamma$  defined by*

$$\begin{aligned} \tau_K(A_0, A_1, A_2) &= \sum_{j=1}^g \kappa c_{j,j+g}(A_0, A_1, A_2) \\ &= \sum_{j=1}^g \int_{X_\Gamma \times X \times X} \kappa \Psi_j(z, x, y) \varpi(z, x, y) k_0(z, x, r) k_1(x, y, r) k_2(y, z, r) dz dx dy \end{aligned}$$

for  $r \in \Omega$ . Here the  $k_j$  are the kernels of the  $A_j, j = 0, 1, 2$  (three exponentially decaying elements of  $\mathcal{B}^\Gamma$ ) and  $\sum_{j=1}^g \Psi_j(z, x, y)$  is proportional to the ‘symplectic area’ of the Euclidean triangle  $\Delta_E$  in  $\mathbb{R}^{2g}$  with vertices

$\Xi(x), \Xi(y), \Xi(z)$ . Here  $\kappa = 4\pi(g-1)/g$  is a constant depending only on the genus  $g$ , where  $g > 1$ .

The constant  $\kappa = 4\pi(g-1)/g$  is justified in the discussion following Theorem 2.3 below. To compare the conductance cocycle  $\tau_K$  with the Chern character cocycle  $\tau_{c,\Gamma}$ , we begin by recalling the following Theorem 5.5.1, page 222 in [10].<sup>1</sup>

**Theorem 2.3.** *Let  $\Sigma$  be a compact Riemann surface of genus  $g \geq 2$  and  $\alpha_1, \dots, \alpha_g$  be a basis of holomorphic 1-forms on  $\Sigma$ . Then  $\sum_{j=1}^g \alpha_j \otimes \bar{\alpha}_j$  defines a Kähler metric on  $\Sigma$  called the Bergman metric or the canonical metric, that has nonpositive curvature vanishing at most at a finite number of points on  $\Sigma$ .*

It follows from this theorem, which uses the Riemann-Roch theorem, that  $\omega_\alpha = \frac{\sqrt{-1}}{2} \sum_{j=1}^g \alpha_j \wedge \bar{\alpha}_j$  is a volume form on  $\Sigma$ . This is a subtle result as the holomorphic 1-form  $\alpha_j$  cannot be nowhere zero, which follows by an application of the Hopf index theorem, where we observe that the Euler characteristic is nonzero. Therefore each term  $\frac{\sqrt{-1}}{2} \alpha_j \wedge \bar{\alpha}_j$  by itself cannot be a volume form on  $\Sigma$ !

Next we recall the following basic fact relating holomorphic 1-forms and harmonic 1-forms on  $\Sigma$ . A (complex valued) 1-form  $\alpha$  on  $\Sigma$  is holomorphic if and only if  $\alpha = a + \sqrt{-1} * a$ , where  $a$  is a (real valued) harmonic 1-form on  $\Sigma$  and  $*a$  is the Hodge  $*$  of  $a$ .

If  $a_j, j = 1, \dots, 2g$  is a symplectic basis of harmonic 1-forms on  $\Sigma$ , where  $a_{j+g} = *a_j, j = 1, \dots, g$ . Then  $\alpha_j = a_j + \sqrt{-1} a_{j+g}$  is a basis of holomorphic 1-forms on  $\Sigma$ . By Theorem 2.3 and its consequence, we deduce that  $\sum_{j=1}^g \alpha_j \wedge \bar{\alpha}_j$  is a volume form on  $\Sigma$ .

Now let  $\omega_\Sigma$  denote the volume form on  $\Sigma = \mathbb{H}/\Gamma$  induced by the hyperbolic volume form  $\omega_{\mathbb{H}}$  on  $\mathbb{H}$ . Then there is a positive constant  $\kappa$  such that  $\omega_\Sigma$  and  $\kappa \sum_{j=1}^g \alpha_j \wedge \bar{\alpha}_j$  are cohomologous. To determine the constant  $\kappa$ , we integrate over the surface  $\Sigma$  to get

$$\int_{\Sigma} \omega_\Sigma = \kappa \int_{\Sigma} \sum_{j=1}^g \alpha_j \wedge \bar{\alpha}_j.$$

Now each term  $\int_{\Sigma} \alpha_j \wedge \bar{\alpha}_j = 1$  by our choice of normalized symplectic basis. By the Gauss-Bonnet theorem  $\int_{\Sigma} \omega_\Sigma = 4\pi(g-1)$ . Therefore  $\kappa = 4\pi(g-1)/g$ .

Thus by the argument above and Lemma 1.9, we see that the difference  $\omega_{\mathbb{H}} - \kappa \Xi^*(\omega_J) = d\Lambda$ , where  $\Lambda$  is a  $\Gamma$ -invariant 1-form on  $\mathbb{H}$ . More particularly

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<sup>1\*</sup> Note that in [4] page 652, we used a different, incorrect argument at this point, and we thank Siye Wu for pointing this out to us.

for a geodesic triangle  $\Delta \subset \mathbb{H}$  with vertices at  $x, y, z \in \mathbb{H}$ ,

$$\begin{aligned} \int_{\Delta} \omega_{\mathbb{H}} &= \kappa \int_{\Delta} \Xi^*(\omega_J) + \int_{\Delta} d\Lambda \\ &= \kappa \int_{\Xi(\Delta)} \omega_J + \int_{\partial\Delta} \Lambda \end{aligned}$$

Now  $\Xi$  cannot map geodesic triangles to Euclidean triangles in  $\mathbb{R}^{2g}$  as  $\Xi(\Delta)$  is a compact subset of a non-flat embedded two dimensional surface in  $\mathbb{R}^{2g}$ . Moreover as  $\Psi_j(z, x, y) = 0$  whenever the images of  $z, x, y$  under  $\Xi$  lie in a Lagrangian subspace (with respect to the symplectic form  $s$ ) of  $\mathbb{R}^{2g}$ ,  $\tau_K$  and  $\tau_{c,\Gamma}$  are not obviously proportional.

Next we write  $\omega_J = d\theta$ . Considering the difference  $\tau_K - \tau_{c,\Gamma}$  one sees that the key is to understand

$$\int_{\Xi(\Delta)} \omega_J - \int_{\Delta_E} \omega_J = \int_{\partial\Xi(\Delta)} \theta - \int_{\partial\Delta_E} \theta.$$

Now this difference of integrals around the boundary can be written as the sum of three terms corresponding to splitting the boundaries  $\partial\Xi(\Delta)$  and  $\partial\Delta_E$  into three arc segments each. We introduce some notation for this, writing

$$\partial\Xi(\Delta) = \Xi(\ell(x, y)) \cup \Xi(\ell(y, z)) \cup \Xi(\ell(z, x)),$$

where  $\ell(x, y)$  is the geodesic in  $\mathbb{H}$  joining  $x$  and  $y$  (with the obvious similar definition of the other terms). We also write

$$\partial\Delta_E = m(x, y) \cup m(y, z) \cup m(z, x),$$

where  $m(x, y)$  is the straight line joining  $\Xi(x)$  and  $\Xi(y)$  (and again the obvious definition of the other terms). Then we have

$$\int_{\partial\Xi(\Delta)} \theta - \int_{\partial\Delta_E} \theta = h(x, y) + h(y, z) + h(z, x) \quad (*)$$

where  $h(x, y) = \int_{\Xi(\ell(x, y))} \theta - \int_{m(x, y)} \theta$  with similar definitions for  $h(y, z)$  and  $h(z, x)$ .

Notice that we have  $h(x, y) = \int_{D_{xy}} \omega_J$  where  $D_{xy}$  is a disc with boundary  $m(x, y) \cup \Xi(\ell(x, y))$ . From this it is easy to see that  $h(\gamma x, \gamma y) = h(x, y)$  for  $\gamma \in \Gamma$ .

Now consider  $j(x, y) = \int_{\ell(x, y)} \Lambda$ . Since  $\Lambda$  is  $\Gamma$ -invariant, it follows that  $j(\gamma x, \gamma y) = j(x, y)$  for  $\gamma \in \Gamma$ . Then by the computation done above, we see that

$$\int_{\Delta} \omega_{\mathbb{H}} = \kappa \int_{\Delta_E} \omega_J + \kappa(h(x, y) + h(y, z) + h(z, x)) + j(x, y) + j(y, z) + j(z, x)$$

We normalise  $\sum_{j=1}^g \Psi_j(z, x, y)$  so that it equals  $\int_{\Delta_E} \omega_J$ . Then,

$$\Phi(x, y, z) = \kappa \sum_{j=1}^g \Psi_j(z, x, y) + \partial(\kappa h + j)(x, y, z)$$

where  $\Phi(x, y, z) = \int_{\Delta} \omega_{\mathbb{H}}$ .

Introduce the bilinear functional  $\tau_1$  on  $\mathcal{B}^\Gamma$  given by

$$\begin{aligned} \tau_1(A_0, A_1) &= \int_{X_\Gamma \times X} (h(x, y) + j(x, y)) k_0(x, y) k_1(y, x) dx dy \\ &= \text{tr}_{\mathcal{B}^\Gamma} (A_{\kappa h + j} A_1), \end{aligned}$$

where the operator  $A_j$  has kernel  $k_j(x, y, r)$ ,  $j = 0, 1$  and  $A_{\kappa h + j}$  is the operator with kernel  $(\kappa h(x, y) + j(x, y)) k_0(x, y, r)$ . So we have proved that formally the two cyclic 2-cocycles satisfy,

$$b\tau_1 = \tau_K - \tau_{c, \Gamma},$$

where  $b$  is the Hochschild boundary operator, so that they are cohomologous cyclic 2-cocycles. What remains is to understand the domain of the cochains, which is what is addressed next.

We want to see that  $\tau_1$  is densely defined. By Theorem 1.5, one has an isomorphism

$$\Phi_F : \mathcal{B}^\Gamma \cong C(\Omega) \rtimes_{\bar{\sigma}} \Gamma \otimes \mathcal{K}(L^2(F)).$$

Here  $F$  denotes a fundamental domain for the action of  $\Gamma_g$  on  $\mathbb{H}$ . Now any element  $x$  in  $C(\Omega) \rtimes_{\bar{\sigma}} \Gamma \otimes \mathcal{K}$  can be written as a matrix  $(x_{ij})$ , where  $x_{ij} \in C(\Omega) \rtimes_{\bar{\sigma}} \Gamma$ . So we can define

$$N_k(x) = \left( \sum_{i,j} \nu(x_{ij})^2 \right)^{\frac{1}{2}},$$

where

$$\nu(x_{ij}) = \left( \sum_{h \in \Gamma_g} (1 + \ell(h)^{2k}) |x(h)|^2 \right)^{\frac{1}{2}}$$

and  $\ell$  denotes the word length function on the group  $\Gamma_g$ . Using a slight modification of the argument given in [8], III.5.γ, one can prove that there is a subalgebra  $\mathcal{B}_\infty^\Gamma$  of  $\mathcal{B}^\Gamma$  which

- (i) contains  $C(\Omega) \rtimes_{\bar{\sigma}, \text{alg}} \Gamma \otimes \mathcal{R}$ , where  $\mathcal{R}$  denotes the algebra of smoothing operators on  $F$  and  $\rtimes_{\bar{\sigma}, \text{alg}}$  denotes the algebraic twisted crossed product,
- (ii) is stable under the holomorphic functional calculus, and
- (iii) is such that  $N_k(x) < \infty$  for all  $x \in \mathcal{B}_\infty^\Gamma$  and  $k \in \mathbb{N}$ .

Then, following [8], we have that the trace  $\tau \otimes \text{Tr}$  on  $C(\Omega) \rtimes_{\bar{\sigma}, \text{alg}} \Gamma \otimes \mathcal{R}$ , is continuous for the norm  $N_k$ , for  $k$  sufficiently large, and thus extends by continuity to  $\mathcal{B}_\infty^\Gamma$ . Note that elements in  $\mathcal{B}_\infty^\Gamma$  have Schwartz kernels which have rapid decay away from the diagonal. The next result summarises the discussion above.

**Proposition 2.4.** *The algebra  $\mathcal{B}_\infty^\Gamma$  is dense in  $\mathcal{B}^\Gamma$ , is closed under the holomorphic functional calculus and is contained in the ideal  $\mathcal{I}$  of  $\mathcal{B}^\Gamma$  consisting of operators with finite trace.*

Now  $\tau_K$  is defined on  $\mathcal{B}_\infty^\Gamma$  while  $\tau_{c,\Gamma}$  is defined on  $\mathcal{B}_0^\Gamma$  as we noted earlier. Both of these algebras contain the operators whose Schwartz kernels are supported in a band around the diagonal. Thus the subalgebra  $\mathcal{B}_\infty^\Gamma \cap \mathcal{B}_0^\Gamma$  is dense and stable under the holomorphic functional calculus. Since  $\Lambda$  is  $\Gamma$ -invariant, it is bounded, therefore  $|j(x, y)| \leq \|\Lambda\|_\infty d(x, y)$ , where  $\|\Lambda\|_\infty$  is the supremum norm of  $\Lambda$  and  $d(x, y)$  is the hyperbolic distance from  $x$  to  $y$ . An explicit expression for  $\theta$  shows that it grows linearly in terms of  $d(x, y)$ , so that  $h(x, y)$  grows at worst like  $d(x, y)^2$ . (for more details, see [4]) Therefore if  $A_0 \in \mathcal{B}_\infty^\Gamma$  then so too does  $A_{\kappa h+j}$ . Hence we have  $\tau_1$  defined on  $\mathcal{B}_\infty^\Gamma \cap \mathcal{B}_0^\Gamma$ . This section has proved our main theorem.

**Theorem 2.5.** *The Connes-Kubo cocycle  $\tau_K$  and the Chern character cocycle  $\tau_{c,\Gamma}$  arising as the Chern class of the Fredholm module  $(F, \mathcal{H} \otimes \mathcal{S})$ , are cohomologous as cyclic cocycles on  $\mathcal{B}_\infty^\Gamma \cap \mathcal{B}_0^\Gamma$ .*

### 3. APPENDIX : ON THE QUANTUM ADIABATIC THEOREM (QAT)

One knows that the (time) evolution determined by a time *independent* Hamiltonian reduces to the spectral theory of the Hamiltonian. The QAT says that the (time) evolution of a slowly varying time *dependent* Hamiltonian reduces to the spectral theory of an associated family of adiabatic Hamiltonians. The setting for the QAT is as follows. Let  $s \rightarrow H(s)$  be a smooth family of Hamiltonians (self-adjoint operators)  $\tau =$  time scale and  $s = t/\tau =$  scaled time. Consider now the *physical evolution*

$$i\partial_t U(t) = H(t/\tau)U(t), \quad U(0) = 1$$

or equivalently

$$(1) \quad i\partial_s U_\tau(s) = \tau H_\tau(s)U_\tau(s), \quad U_\tau(0) = 1.$$

Let  $P(0)$  denote the spectral projection onto a gap in the spectrum of  $H(0)$ , that is we have  $P(0) = \chi_{(-\infty, E]}(H(0))$  where  $E \notin$  spectrum of  $(H(0))$ .

The *adiabatic evolution* is determined by the equation

$$(2) \quad P(s) = U_a(s)P(0)U_a(s)^*, \quad U_a(0) = 1$$

where  $P(s)$  denotes spectral projection onto a gap in the spectrum of  $H(s)$ . Let  $H_a(s)$  denote the generator of  $U_a(s)$ . It is also known as the *adiabatic Hamiltonian* and is given by

$$(3) \quad H_a(s) = \frac{i}{\tau}(\partial_s U_a(s))U_a(s)^*$$

**Lemma 3.1.** *The adiabatic Hamiltonian  $H_a(s)$  satisfies the equation of motion*

$$[H_a(s), P(s)] = \frac{i}{\tau} \partial_s P(s)$$

*Proof.* Differentiating (2), we have

$$\begin{aligned} \partial_s P(s) &= \partial_s U_a(s) P(0) U_a(s)^* + U_a(s) P(0) \partial_s U_a(s)^* \\ &= \partial_s U_a(s) P(0) U_a(s)^* - U_a(s) P(0) U_a(s)^* \partial_s U_a(s) U_a(s)^* \\ &= (\partial_s U_a(s)) U_a(s)^* U_a(s) P(0) U_a(s)^* - \frac{\tau}{i} P(s) H_a(s) \\ &= \frac{\tau}{i} [H_a(s), P(s)] \end{aligned}$$

**Lemma 3.2.** *Let  $f$  be a measurable function on  $\mathbb{R}$ . Then  $H_a(s) = f(H(s)) + \frac{i}{\tau} [\partial_s P(s), P(s)]$  satisfies the equations of motion.*

*Proof.*  $[f(H(s)), P(s)] \equiv 0$  and  $[[\partial_s P(s), P(s)], P(s)] = \partial_s P(s)$  since  $P(s)^2 = P(s)$  and  $P(s)$  is a spectral projection of  $H(s)$ . Define the adiabatic Hamiltonian as

$$(4) \quad H_a(s) = H(s) + \frac{i}{\tau} [\partial_s P(s), P(s)]$$

Then equation (2) is satisfied and  $U_a(s) : \text{Range } P(0) \rightarrow \text{Range } P(s)$  i.e. the initial value problem

$$i \partial_s \psi(s) = \tau H_a(s) \psi(s), \quad \psi(0) \in \text{Range } P(0)$$

has the property that  $\psi(s) \in \text{Range } P(s) \forall s$ .

**Theorem 3.3** (Quantum Adiabatic Theorem (QAT) [1]). *Let  $s \rightarrow H(s)$  be a smooth family of self-adjoint Hamiltonians and  $s \rightarrow P(s)$  be a smooth family of spectral projections as before such that*

$$\sup\{\|P(s)\| < \infty \mid s \in [0, \infty)\}$$

*and the commutator equation  $[\partial_s P(s), P(s)] = [H(s), X(s)]$  has an operator-valued solution  $X(s)$ , such that  $X(s)$  and  $\partial_s X(s)$  are bounded. Then one has*

$$\|(U_\tau(s) - U_a(s))P(0)\| \leq \frac{1}{\tau} \max_{s \in [0, \infty)} \{2\|X(s)P(s)\| + \|\partial_s(X(s)P(s))P(s)\|\}$$

*That is, the adiabatic evolution  $U_a(s)$  approximates the physical evolution  $U_\tau(s)$  as the adiabatic parameter  $\tau \rightarrow \infty$ . Equivalently, the adiabatic Hamiltonian  $H_a(s)$  approximates the physical Hamiltonian  $H_\tau(s)$  on the range of  $P$ , as the adiabatic parameter  $\tau \rightarrow \infty$ .*

Note that the hypotheses on  $P(s)$  are satisfied if  $P(s)$  is a spectral projection onto a gap in the spectrum of  $H(s)$  because one can then define

$$X(s) = \frac{1}{2\pi i} \oint_C R(z, s) \partial_s P(s) R(z, s) dz$$

where  $C$  is a contour in  $\mathbb{C}$  enclosing the spectrum in  $(-\infty, E]$ ,  $E \notin \text{spec}(H(s))$  and  $R(z, s) = (H(s) - z)^{-1}$  is the resolvent.

#### 4. APPENDIX: CONDUCTANCE COCYCLES

In this subsection we present an argument which derives from physical principles the hyperbolic Connes-Kubo formula for the ‘Hall conductance’. Our reasoning is that the Hall conductance in the Euclidean situation is measured experimentally by determining the equilibrium ratio of the current in the direction of the applied electric field to the Hall voltage, which is the potential difference in the orthogonal direction. To calculate this mathematically we instead determine the component of the induced current that is orthogonal to the applied potential. The conductance can then be obtained by dividing this quantity by the magnitude of the applied field. In the hyperbolic case preferred directions are obtained by interpreting the generators of the fundamental group as geodesics on hyperbolic space giving a family of preferred directions emanating from the base point. For each pair of directions it is therefore natural to imitate the procedure of the Euclidean case and mathematically this is done as follows.

The Hamiltonian  $H$  in a magnetic field depends on the magnetic vector potential  $\mathbf{A}$  and the functional derivative  $\delta_k H$  of  $H$  with respect to one of the components of  $\mathbf{A}$ , denoted  $A_k$ , gives the current density  $J_k$ , where we consider adiabatic variations within a one-parameter family  $A_k(s)$ , which we can choose without loss of generality to be bounded, since  $\mathbf{A}(0) = -\theta \frac{dx}{y}$  defines a bounded operator in the hyperbolic metric. The expected value of the current in a state described by a projection operator  $P$  into a spectral gap of  $H$  is therefore  $\text{tr}(P \delta_k H)$  (cf [1] equation (3.2)). (Note that an argument, using the fact that  $P$  is a member of a family  $P(s)$  of projections which correspond to gaps for small  $s$ , is required to see that  $P \delta_k H$  is trace class.) The following lemma is not proved by a rigorous argument: one needs to check various analytical details as in [20] which we omit as they would take us too far afield. For this discussion  $\text{tr}$  will denote a generic trace.

**Lemma 4.1.** *In the adiabatic limit as the adiabatic parameter  $\tau \rightarrow \infty$ , the functional derivative of the adiabatic Hamiltonian  $\delta_k H_a(s)$  approximates the functional derivative of the physical Hamiltonian  $\delta_k H_\tau(s)$  on the range of  $P$ , and one has*

$$\text{tr}(P \delta_k H) = i \text{tr}(P [\partial_t P, \delta_k P]).$$



*Proof.* The first statement is the result of a calculation. It uses the explicit forms of  $\delta_k$  and  $H_a(s)$  and the fact that the family  $A_k(s)$  is bounded to show that the norm of the difference  $\delta_k H_a(s) - \delta_k H_\tau(s)$  goes to zero as  $s \rightarrow 0$ . By using the invariance of the trace under the adjoint action of operators and the equation of motion we see that

$$\begin{aligned} \operatorname{tr}(P[\partial_t P, \delta_k P]) &= -\operatorname{tr}([P, \delta_k P]\partial_t P) \\ &= -i \operatorname{tr}([P, \delta_k P][P, H_a]) \\ &= i \operatorname{tr}([P, [P, \delta_k P]]H_a). \end{aligned}$$

Now  $\delta_k P = \delta_k(P^2) = P(\delta_k P) + (\delta_k P)P$ , whence  $P(\delta_k P)P = 0$  and we have

$$\begin{aligned} [P, [P, \delta_k P]] &= P(P(\delta_k P) - (\delta_k P)P) - (P(\delta_k P) - (\delta_k P)P)P \\ &= P(\delta_k P) + (\delta_k P)P = \delta_k P. \end{aligned}$$

Consequently we may write

$$\operatorname{tr}(P[\partial_t P, \delta_k P]) = i \operatorname{tr}((\delta_k P)H_a) = i \operatorname{tr}(\delta_k(PH_a)) - i \operatorname{tr}(P(\delta_k H_a)),$$

and, assuming that the trace is invariant under variation of  $A_k$ , the first term vanishes. The result asserted follows by taking the limit as the adiabatic parameter  $\tau \rightarrow \infty$ . By following [11], one sees that in fact the limit of the lemma is true to all orders. We note further that if the only  $t$ -dependence in  $H$  and  $P$  is due to the adiabatic variation of  $A_j$ , a component distinct from  $A_k$ , then  $\partial_t = \partial A_j / \partial t \times \delta_j$ . Working in the Landau gauge so that the electrostatic potential vanishes, the electric field is given by  $\mathbf{E} = -\partial \mathbf{A} / \partial t$ , and so  $\partial_t = -E_j \delta_j$ . Combining this with the previous argument we arrive at the following result:

**Corollary 4.2.** *The conductance for currents in the  $k$  direction induced by electric fields in the  $j$  direction is given by  $-i \operatorname{tr}(P[\delta_j P, \delta_k P])$ .*

*Proof.* The expectation of the current  $J_k$  is given by

$$\operatorname{tr}(P\delta_k H) = i \operatorname{tr}(P[\partial_t P, \delta_k P]) = -iE_j \operatorname{tr}(P[\delta_j P, \delta_k P]),$$

from which the result follows immediately.

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